arXiv: 1003.3390 [hep-th]
CAS-PHYS-BHU Preprint

Absolutely anticommuting (anti-)BRST symmetry transformations for topologically massive Abelian gauge theory

S. Gupta[(a)], R. Kumar[(a)], R. P. Malik[(a,b)]

[(a)] *Physics Department, Centre of Advanced Studies, Banaras Hindu University, Varanasi - 221 005, (U.P.), India*

and

[(b)] *DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi - 221 005, India*

e-mails: guptasaurabh4u@gmail.com, raviphynuc@gmail.com, malik@bhu.ac.in

Abstract: We demonstrate the existence of the nilpotent and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for the four (3 + 1)-dimensional (4D) topologically massive Abelian $U(1)$ gauge theory that is described by the coupled Lagrangian densities (which incorporate the celebrated $(B \wedge F)$ term). The absolute anticommutativity of the (anti-) BRST symmetry transformations is ensured by the existence of a Curci-Ferrari type restriction that emerges from the superfield formalism as well as from the equations of motion that are derived from the above coupled Lagrangian densities. We show the invariance of the action from the point of view of the symmetry considerations as well as superfield formulation. We discuss, furthermore, the topological term within the framework of superfield formalism and provide the geometrical meaning of its invariance under the (anti-) BRST symmetry transformations.

PACS numbers: 11.15.Wx, 11.15.-q, 03.70.+k, 12.90.+b

Keywords: Topologically massive Abelian $U(1)$ gauge theory in 4D; nilpotency and absolute anticommutativity; (anti-) BRST symmetry transformations; superfield formulation; geometrical interpretations

1. Introduction

A couple of decisive mathematical features, that are closely connected with the basic concepts of Becchi-Rouet-Stora-Tyutin (BRST) formalism [1-4], are

(i) the nilpotency ($s_{(a)b}^2 = 0, Q_{(a)b}^2 = 0$) of the (anti-) BRST symmetry transformations ($s_{(a)b}$) and their corresponding generators ($Q_{(a)b}$), and

(ii) the absolute anticommutativity ($s_b s_{ab} + s_{ab} s_b = 0,\ Q_b Q_{ab} + Q_{ab} Q_b = 0$) of the (anti-) BRST symmetry transformations $s_{(a)b}$ (in their operator form) and the generators $Q_{(a)b}$ which generate the transformations $s_{(a)b}$.

The former mathematical property physically implies the fermionic nature of $s_{(a)b}$ (as well as $Q_{(a)b}$) and the latter property encodes the linear independence of s_b *vis-à-vis* s_{ab} (and Q_b versus Q_{ab}). These mathematical properties are very sacrosanct and they must be obeyed in the BRST description of any arbitrary gauge/reparametrization invariant theories.

The role of the BRST formalism is quite significant in the description of the non-Abelian 1-form gauge theories which are at the heart of theoretical foundations of the standard model of high energy physics. Despite stunning success stories associated with the standard model, its shortcomings are the detection of the mass of the neutrino and no experimental evidence for the Higgs boson (so far!). One of the roles of the Higgs particle is to generate suitable masses for the gauge particles and fermions. Thus, its detection is very crucial for the sanctity of the theoretical foundations of the standard model. Since the esoteric Higgs bosons have not yet been seen experimentally, some alternative models have been proposed for the mass generation, symmetry breaking, etc. One of the alternate models, for the mass generation of the gauge fields, is the inclusion of the topological ($B \wedge F$) term in the Lagrangian density of the 1-form and 2-form (non-) Abelian gauge theories where the mass generation of the 1-form gauge boson is very natural [5-8].

The 2-form [$B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu) B_{\mu\nu}$] antisymmetric tensor gauge field $B_{\mu\nu}$ [9,10] has become quite popular because of its relevance in the context of superstring [11,12] and supregravity theories [13]. Besides being a theoretical generalization of the 1-form gauge field [14], it provides the field theoretic models for the Hodge theory [15-17] and it is also relevant in the context of condensed matter physics [18]. Its constraint structures [19], BRST quantization scheme [20-22], etc., have been studied. These studies have led to some novel features (that are found to be *absent* in the study of 1-form (non-) Abelian gauge theories). Thus, the (non-) Abelian 2-form gauge field is endowed with a very rich mathematical and theoretical structures.

In our present endeavor, we focus on the gauge theory of the Abelian 1-form and 2-form gauge fields that are coupled with each-other through the

famous $(B \wedge F)$ term. In fact, in the presence of the $(B \wedge F)$ term, we study the present (4D topological massive Abelian $U(1)$ gauge) model within the framework of BRST formalism. We promote the gauge symmetry of the theory to the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations where the basic tenets of the BRST formalism are fully respected. We find some novel features in our study. These are

(i) the existence of the coupled Lagrangian densities for the description of an Abelian gauge theory that incorporates the Abelian 1-form and 2-form gauge fields together with the topological term. This observation is novel in the sense that it is very similar to the case of the non-Abelian 1-form gauge theory where such a kind of Lagrangian densities do exist [23,24],

(ii) the derivation of the Curci-Ferrari (CF) type restriction from the coupled Lagrangian densities as well as from the superfield approach to BRST formalism. This aspect of our present Abelian theory is exactly same as the one observed in the case of non-Abelian 1-form gauge theory [23,24] (where, for the first time, CF condition appeared [25]), and

(iii) the interpretation of the topological $(B \wedge F)$ term within the framework of the superfield approach to BRST formalism and its geometrical meaning *vis-à-vis* the rest of the terms of the theory.

The key factors that have contributed to our main motivation for present investigation are as follows. First and foremost, the BRST construction, for our present model, has been found to be endowed with the BRST symmetries that are non-nilpotent (see, e.g. [26]). Thus, it is an interesting endeavor for us to obtain the symmetries that obey the key requirements of the BRST formalism. Second, we demonstrate that our present model is described by a coupled set of Lagrangian densities due to the existence of CF-type restriction. Third, it is important for us to check the relevance of our earlier work [27] in the context of our present model which is more general than the BRST description of the *free* Abelian 2-form gauge theory. Fourth, our present 4D theory provides a field theoretic model where the superfield and Lagrangian approaches to the BRST formalism blend together in a useful and clarifying manner. Finally, the *non-Abelian* generalization of the present model has been a topic of intense research for quite sometime [8,28,29]. We wish to generalize our present model to the non-Abelian case by exploiting the superfield formalism proposed by Bonora, etal. [30,31].

Our present paper is organized as follows. In Sec. 2, we discuss about the gauge symmetries and constraint structures of the 4D massive Abelian $U(1)$ gauge theory. Our Sec. 3 is devoted to the discussion of the on-shell nilpotent BRST and anti-BRST symmetry transformations for a *single* Lagrangian density and we demonstrate that these transformations are non-anticommuting in nature. In Sec. 4, we provide a brief synopsis of the

superfield approach to derive the proper and precise (anti-) BRST symmetry transformations. In Secs. 5 and 6, we dwell a bit on the (anti-) BRST invariance of the present theory within the frameworks of the Lagrangian and superfield formalisms, respectively. Finally, we make some concluding remarks and point out a few future directions in Sec. 7.

2. Preliminaries: gauge symmetries and constraints

We begin with the Lagrangian density of a massive gauge invariant Abelian model in four (3 + 1)-dimensions of spacetime. This Lagrangian density incorporates the celebrated topological ($B \wedge F$) term as given below[1]

$$\mathcal{L}_0 = -\,\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{1}{12}\,H^{\mu\nu\eta}\,H_{\mu\nu\eta} + \frac{m}{4}\,\varepsilon^{\mu\nu\eta\kappa}\,F_{\mu\nu}\,B_{\eta\kappa}, \tag{1}$$

where the totally antisymmetric quantities $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $H_{\mu\nu\eta} = \partial_\mu B_{\nu\eta} + \partial_\nu B_{\eta\mu} + \partial_\eta B_{\mu\nu}$ are the curvature tensors owing their origin to 2-form $F^{(2)} = dA^{(1)} \equiv \frac{1}{2!}\,(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ and 3-form $H^{(3)} = dB^{(2)} \equiv \frac{1}{3!}\,(dx^\mu \wedge dx^\nu \wedge dx^\eta) H_{\mu\nu\eta}$, respectively. Here $d = dx^\mu\,\partial_\mu$ (with $d^2 = 0$) is the exterior derivative and 1-form $A^{(1)} = dx^\mu A_\mu$ and 2-form $B^{(2)} = \frac{1}{2!}\,(dx^\mu \wedge dx^\nu)\,B_{\mu\nu}$ define the Abelian 4-vector (A_μ) and second rank anti-symmetric ($B_{\mu\nu} = -B_{\nu\mu}$) tensor ($B_{\mu\nu}$) gauge fields. It is clear that the parameter 'm' has the dimension of mass in the physical four dimensions of spacetime.

The above Lagrangian density transforms to a total spacetime derivative (i.e. $\delta_{(gt)}\,\mathcal{L}_0 = \partial_\mu[m\,\varepsilon^{\mu\nu\eta\kappa}\Lambda_\nu\,(\partial_\eta A_\kappa)]$) under the following infinitesimal gauge transformations[2] $\delta_{(gt)}$ (with gauge parameter Λ and Λ_μ)

$$\delta_{(gt)} A_\mu = \partial_\mu \Lambda, \qquad\qquad \delta_{(gt)} B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{2}$$

Thus, the action $S_0 = \int d^4x\;\mathcal{L}_0$ remains invariant under the infinitesimal gauge transformations (2). It is straightforward to check that the following Euler-Lagrange equations of motion

$$\partial_\mu F^{\mu\nu} = \frac{1}{2}\,m\,\varepsilon^{\mu\nu\eta\kappa}\partial_\mu B_{\eta\kappa}, \qquad \partial_\mu H^{\mu\nu\eta} = \frac{1}{2}\,m\,\varepsilon^{\nu\eta\kappa\sigma} F_{\kappa\sigma}, \tag{3}$$

[1]We adopt here the flat metric ($\eta_{\mu\nu}$) with signatures $(+1, -1, -1, -1)$ so that $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i$ is the dot product between two non-null vectors P_μ and Q_μ where $\mu, \nu, \eta, \kappa, ... = 0, 1, 2, 3$ correspond to the spacetime directions and $i, j, k, ... = 1, 2, 3$ stand for the space directions only. We make the choice $\varepsilon_{0123} = +1 = -\varepsilon^{0123}$ for the totally antisymmetric Levi-Civita tensor ($\varepsilon_{\mu\nu\eta\kappa}$) that obeys $\varepsilon_{\mu\nu\eta\kappa}\varepsilon^{\mu\nu\eta\kappa} = -4!$, $\varepsilon_{\mu\nu\eta\kappa}\varepsilon^{\mu\nu\eta\sigma} = -3!\;\delta^\sigma_\kappa$, etc. The component $\varepsilon_{0ijk} = \epsilon_{ijk}$ is the 3D Levi-Civita tensor.

[2]Note that the Lagrangian density $\mathcal{L}_0$ respects a couple of independent gauge symmetry transformations: (i) $A_\mu \to A'_\mu = A_\mu,\; B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu} + (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu)$, and (ii) $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda,\; B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu}$. For the sake of generality, however, we have taken the combination of these two transformations *together* in (2).

emerge from the Lagrangian density (1). The components of the conjugate momenta with respect to the vector field A_μ and tensor field $B_{\mu\nu}$:

$$
\begin{aligned}
\Pi^0_{(A)} &= 0, \qquad\qquad & \Pi^i_{(A)} &= -F^{0i} + \frac{1}{2}\, m\, \varepsilon^{0ijk} B_{jk}, \\
\Pi^{0i}_{(B)} &= 0, \qquad\qquad & \Pi^{ij}_{(B)} &= \frac{1}{2} H^{0ij},
\end{aligned} \tag{4}
$$

ensure that $\Pi^0_{(A)} \approx 0,\ \Pi^{0i}_{(B)} \approx 0$ are the primary constraints on the theory. As a consequence, the equations of motion with respect to A_0 field and B_{0i} field (see, e.g. [32] for details):

$$
\begin{aligned}
\partial_i \Big(F_{0i} - \frac{1}{2}\, m\, \epsilon_{ijk} B_{jk} \Big) &\equiv -\, \partial_i \Pi^{(A)}_i \approx 0, \\
\partial_j H_{0ij} + \frac{1}{2}\, m\, \epsilon_{ijk} F_{jk} &\equiv 2\, \partial_j \Pi^{(B)}_{ij} + \frac{1}{2}\, m\, \epsilon_{ijk} F_{jk} \approx 0,
\end{aligned} \tag{5}
$$

lead to the derivation of the secondary constraints on the theory. The above primary and secondary constraints are the first-class constraints in the language of Dirac's prescription for the classification scheme [33,34].

The continuous gauge symmetry transformations (2) lead to the derivation of the Noether conserved current as given below:

$$
J^\mu_{(gt)} = \frac{1}{2}\, m\, \varepsilon^{\mu\nu\eta\kappa}\, (\partial_\nu \Lambda) B_{\eta\kappa} - F^{\mu\nu}\, \partial_\nu \Lambda + H^{\mu\nu\eta}\, \partial_\nu \Lambda_\eta - m\, \varepsilon^{\mu\nu\eta\kappa} \Lambda_\nu\, (\partial_\eta A_\kappa), \tag{6}
$$

because $\partial_\mu J^\mu_{(gt)} = 0$ when we exploit the Euler-Lagrange equations of motion (3). The conserved charge (i.e. $Q_{(gt)} = \int d^3x\ J^0_{(gt)}$)

$$
\begin{aligned}
Q_{(gt)} &= \int d^3x \Big[\Big(F_{0i} - \frac{1}{2}\, m\, \epsilon_{ijk} B_{jk} \Big) \partial_i \Lambda + 2\, \Pi^{ij}_{(B)} (\partial_i \Lambda_j) + m\, \epsilon_{ijk}\, \Lambda_i\, (\partial_j A_k) \Big] \\
&\equiv \int d^3x \Big[\Pi^i_{(A)} (\partial_i \Lambda) + 2\, \Pi^{ij}_{(B)} (\partial_i \Lambda_j) + m\, \epsilon_{ijk}\, \Lambda_i\, (\partial_j A_k) \Big],
\end{aligned} \tag{7}
$$

generates the following transformations with the help of (11) (see below)

$$
\begin{aligned}
\delta_{(gt)} A_i &= -i\, [A_i,\ Q_{(gt)}] = \partial_i \Lambda, \\
\delta_{(gt)} B_{ij} &= -i\, [B_{ij},\ Q_{(gt)}] = \partial_i \Lambda_j - \partial_j \Lambda_i.
\end{aligned} \tag{8}
$$

Thus, the Noether conserved charge $Q_{(gt)}$ does *not* generate all the transformations for *all* the components of the field. For instance, we can never be able to obtain the transformations for the components A_0 and B_{0i} of the 1-form and 2-form gauge fields, respectively, from the above charge $Q_{(gt)}$.

The basic tenet of gauge theory ensures that all the gauge transformations should be generated by the first-class constraints of the theory [35]. Such, a generator (G), in terms of the above first-class constraints, is[3]

$$
\begin{aligned}
G &= \int d^3x \,\Big[(\partial_0 \Lambda)\, \Pi^0_{(A)} + \Lambda\, \partial_i \Pi^i_{(A)} + (\partial_0 \Lambda_i - \partial_i \Lambda_0)\, \Pi^{0i}_{(B)} \\
&+ (\partial_i \Lambda_j - \partial_j \Lambda_i)\, \Pi^{ij}_{(B)}\Big]. \qquad (9)
\end{aligned}
$$

The above generator leads to the derivation of (2) if we exploit the following general rule for the transformation of the generic field Φ, namely;

$$
\delta_{(gt)} \Phi = -i\, \big[\Phi,\, G\big], \qquad \Phi = A_\mu, B_{\mu\nu}, \qquad (10)
$$

supplemented with the following canonical commutation relations

$$
\begin{aligned}
\big[A_0(\mathbf{x}, t),\, \Pi^0_{(A)}(\mathbf{y}, t)\big] &= i\, \delta^{(3)}(\mathbf{x} - \mathbf{y}), \\
\big[A_i(\mathbf{x}, t),\, \Pi^j_{(A)}(\mathbf{y}, t)\big] &= i\, \delta^j_i\, \delta^{(3)}(\mathbf{x} - \mathbf{y}), \\
\big[B_{0i}(\mathbf{x}, t),\, \Pi^{0j}_{(B)}(\mathbf{y}, t)\big] &= i\, \delta^j_i\, \delta^{(3)}(\mathbf{x} - \mathbf{y}), \\
\big[B_{ij}(\mathbf{x}, t),\, \Pi^{kl}_{(B)}(\mathbf{y}, t)\big] &= \frac{i}{2}\, (\delta^k_i\, \delta^l_j - \delta^l_i\, \delta^k_j)\, \delta^{(3)}(\mathbf{x} - \mathbf{y}), \qquad (11)
\end{aligned}
$$

and all the rest of the brackets should be taken to be zero.

At this stage, a couple of key points are to be noted. First, neither the conserved charge $Q_{(gt)}$ nor the generator G produces the residual symmetry transformation[4] that is present in the gauge transformations $\delta_{(gt)} B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ when $\Lambda_\mu \to \Lambda_\mu + \partial_\mu \omega$. Second, according to the Dirac's prescription for the quantization of system with constraints, we must demand that the physical states of the theory should be annihilated by the first-class constraints (and the ensuing conditions should remain invariant with respect to the time evolution of the system). We do not obtain these conditions from $Q_{(gt)}$ and G (unless we impose the same, by hand, from outside).

The resolutions of these important issues could be addressed within the framework of BRST formalism. This is what precisely we envisage to do in our forthcoming sections. We also comment on various subtle issues that

[3]It will be noted that one of the secondary constraints (cf. (5)) includes the topological term "$m\epsilon_{ijk}F_{jk}$" as well. However, this term does not generate any transformation. Thus, we have not incorporated this term in the expression for G so that we could get a compact and simple form of G. In principle, this term should be present in our expression for G.

[4]In other words, if we assume that the gauge parameter Λ_μ is a field that transforms as $\delta_\omega\, \Lambda_\mu = \partial_\mu \omega$ under a residual gauge transformation δ_ω, then also, the Lagrangian density remains invariant. We shall see later that the parameter Λ_μ would be identified with the (anti-) ghost fields $(\bar{C}_\mu) C_\mu$ within the framework of BRST formalism (see, Sec. 3).

are associated with the BRST and superfield formulation of the topologically massive Abelian model which is under consideration in our present endeavor.

3. On-shell nilpotent (anti-) BRST invariant Lagrangian density and comments on the covariant canonical quantization

To answer the above raised issues, we begin with a generalized version of Lagrangian density $\mathcal{L}_0$ which incorporates the gauge-fixing terms (in the Feynman gauge) and Faddeev-Popov ghost terms as [26]

$$\begin{aligned}
\mathcal{L}_b &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{12}H^{\mu\nu\eta}H_{\mu\nu\eta} + \frac{1}{4}\, m\, \varepsilon^{\mu\nu\eta\kappa}B_{\mu\nu}F_{\eta\kappa} - \frac{1}{2}(\partial\cdot A)^2 \\
&- \frac{1}{2}(\partial^\nu B_{\nu\mu} - \partial_\mu\phi)^2 - \frac{1}{2}(\partial\cdot\bar{C})(\partial\cdot C) - i\,\partial_\mu\bar{C}\,\partial^\mu C \\
&- (\partial_\mu\bar{C}_\nu - \partial_\nu\bar{C}_\mu)\,(\partial^\mu C^\nu) + \partial_\mu\bar{\beta}\,\partial^\mu\beta, \qquad (12)
\end{aligned}$$

where the fermionic (anti-) ghost fields $(\bar{C}_\mu)C_\mu$ with $(\bar{C}_\mu^2 = C_\mu^2 = 0, C_\mu\bar{C}_\nu + \bar{C}_\nu C_\mu = 0, C_\mu C_\nu + C_\nu C_\mu = 0$, etc.) are the generalization of the gauge parameter Λ_μ and the bosonic (anti-) ghost fields $(\bar{\beta})\beta$ are the generalization of the gauge parameter 'ω' (that was present in the symmetry transformation $\Lambda_\mu \to \Lambda_\mu + \partial_\mu\omega$). In exactly similar fashion, the gauge parameter Λ has been replaced by the fermionic $(C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0)$ (anti-) ghost fields $(\bar{C})C$. It is self-evident that $(\bar{C}_\mu)C_\mu$ and $(\bar{C})C$ have ghost number equal to $(-1)+1$ and $(\bar{\beta})\beta$ have ghost number $(-2)+2$, respectively.

The gauge-fixing term $(\partial^\nu B_{\nu\mu})$ for the 2-form gauge field has its origin in the co-exterior derivative $\delta = -*\ d\ *$ where $*$ is the Hodge duality operation on the 4D spacetime manifold. It can be readily checked that: $\delta B^{(2)} = -*\ d * B^{(2)} = (\partial^\nu B_{\nu\mu})\, dx^\mu$ (i.e. a 1-form). There is a room, however, for adding/subtracting a 1-form to this. This can be constructed with a massless $(\Box\phi = 0)$ scalar field (ϕ) by exploiting an exterior derivative (i.e. $F^{(1)} = dx^\mu\, \partial_\mu\phi$) (see, e.g. [17] for details). This has been done in the above with a minus sign for algebraic convenience. It serves the purpose of stage-one reducibility in the theory (which was *not* incorporated in[5] [26]). The gauge-fixing terms for 1-form (anti-) ghost fields $(\bar{C}_\mu)C_\mu$ as well as Abelian $U(1)$ gauge field A_μ have been taken into account in $\mathcal{L}_b$ by incorporating $(-\frac{1}{2}\,\big(\partial\cdot\bar{C})(\partial\cdot C)\big)$ and $(-\frac{1}{2}\,\big(\partial\cdot A)^2\big)$ terms, respectively. These (with ghost number zero), too, owe their origin to the co-exterior derivative $\delta = -*d*$.

The above Lagrangian density $\mathcal{L}_b$ respects the following nilpotent $\tilde{s}_{(a)b}^2 = 0$ (anti-) BRST symmetries $(\tilde{s}_{(a)b})$ on the on-shell $\big(\Box\beta = 0,\ \Box\bar{\beta} = 0,\ \Box C_\mu -$

[5]It is precisely because of this reason that the BRST transformations, quoted in [26], are *not* on-shell nilpotent of order two.

$\frac{1}{2}\partial_\mu(\partial\cdot C) = 0,\ \Box\bar{C}_\mu - \frac{1}{2}\partial_\mu(\partial\cdot\bar{C}) = 0,\ \Box\phi = 0\big)$. The explicit form of these transformations (as operators on the fields) are

$$
\begin{aligned}
&\tilde{s}_b A_\mu = \partial_\mu C, \quad \tilde{s}_b \bar{C} = -\, i\,(\partial\cdot A), \quad \tilde{s}_b B_{\mu\nu} = (\partial_\mu C_\nu - \partial_\nu C_\mu), \\
&\tilde{s}_b C_\mu = \partial_\mu \beta, \quad \tilde{s}_b \bar{C}_\mu = (\partial^\nu B_{\nu\mu} - \partial_\mu \phi), \quad \tilde{s}_b \phi = -\,\frac{1}{2}\,(\partial\cdot C), \\
&\tilde{s}_b \bar{\beta} = +\,\frac{1}{2}\,(\partial\cdot\bar{C}), \quad \tilde{s}_b C = 0, \quad \tilde{s}_b \beta = 0,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\tilde{s}_{ab} A_\mu = \partial_\mu \bar{C}, \quad \tilde{s}_{ab} C = +\, i\,(\partial\cdot A), \quad \tilde{s}_{ab} B_{\mu\nu} = (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \\
&\tilde{s}_{ab} \bar{C}_\mu = \partial_\mu \bar{\beta}, \quad \tilde{s}_{ab} C_\mu = -\,(\partial^\nu B_{\nu\mu} - \partial_\mu \phi), \quad \tilde{s}_{ab} \phi = -\,\frac{1}{2}\,(\partial\cdot\bar{C}), \\
&\tilde{s}_{ab} \beta = -\,\frac{1}{2}\,(\partial\cdot C), \quad \tilde{s}_{ab} \bar{C} = 0, \quad \tilde{s}_{ab} \bar{\beta} = 0.
\end{aligned}
\tag{14}
$$

In fact, it can be checked that the Lagrangian density $\mathcal{L}_b$ transforms (to the total spacetime derivatives) under the above transformations as

$$
\begin{aligned}
\tilde{s}_b \mathcal{L}_b &= -\,\partial_\mu \Big[(\partial^\mu C^\nu - \partial^\nu C^\mu)(\partial^\sigma B_{\sigma\nu} - \partial_\nu \phi) + \frac{1}{2}\,(\partial_\nu B^{\nu\mu} - \partial^\mu \phi)(\partial\cdot C) \\
&+ (\partial\cdot A)\,\partial^\mu C - \frac{1}{2}\,(\partial\cdot\bar{C})\,\partial^\mu \beta - m\,\varepsilon^{\mu\nu\eta\kappa}\,C_\kappa(\partial_\eta A_\kappa)\Big],
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\tilde{s}_{ab} \mathcal{L}_b &= -\,\partial_\mu \Big[(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial^\sigma B_{\sigma\nu} - \partial_\nu \phi) + \frac{1}{2}\,(\partial_\nu B^{\nu\mu} - \partial^\mu \phi)(\partial\cdot\bar{C}) \\
&+ (\partial\cdot A)\,\partial^\mu \bar{C} + \frac{1}{2}\,(\partial\cdot C)\,\partial^\mu \bar{\beta} - m\,\varepsilon^{\mu\nu\eta\kappa}\,\bar{C}_\kappa(\partial_\eta A_\kappa)\Big].
\end{aligned}
\tag{16}
$$

As a consequence, the action $S = \int d^4x\ \mathcal{L}_b$ remains invariant under the nilpotent symmetry transformations $\tilde{s}_{(a)b}$.

We close this section with the following remarks. First, using the Noether's theorem, one can compute the (anti-) BRST charges $\tilde{Q}_{(a)b}$ which turn out to be conserved and nilpotent. Second, the physicality criteria $\tilde{Q}_{(a)b}|\ phys >= 0$ lead to the annihilation of the physical states $|\ phys >$ by the operator form of the first-class constraints (4) and (5). Third, the analogue of the gauge transformations (2) and residual gauge transformations ($\delta_\omega \Lambda_\mu = \partial_\mu \omega \Rightarrow \tilde{s}_b C_\mu = \partial_\mu \beta,\ \tilde{s}_{ab} \bar{C}_\mu = \partial_\mu \bar{\beta}$) are generated by the nilpotent and conserved charges $\tilde{Q}_{(a)b}$. Fourth, it can be checked that each basic field of the theory has its corresponding canonical momentum. As a consequence, one can perform the *covariant* canonical quantization of the theory in a straightforward manner. Finally, despite the above cited good features, it can be checked that the

above symmetry transformations do not satisfy one of the key decisive requirements of the (anti-) BRST symmetry transformations (connected with a gauge transformation) because the following is *not* true, namely;

$$(\tilde{s}_b\, \tilde{s}_{ab} + \tilde{s}_{ab}\, \tilde{s}_b)\, \Psi = 0 \qquad \Psi = A_\mu,\ C,\ \bar{C},\ B_{\mu\nu},\ C_\mu,\ \bar{C}_\mu,\ \beta,\ \bar{\beta},\ \phi, \tag{17}$$

for the generic field Ψ of the theory. For instance, it can be explicitly checked that we have the following relationships, namely;

$$\begin{aligned}(\tilde{s}_b\, \tilde{s}_{ab} + \tilde{s}_{ab}\, \tilde{s}_b)\, C_\mu\ &= -\ \Box C_\mu \neq 0,\\ (\tilde{s}_b\, \tilde{s}_{ab} + \tilde{s}_{ab}\, \tilde{s}_b)\, \bar{C}_\mu\ &= +\ \Box \bar{C}_\mu \neq 0.\end{aligned} \tag{18}$$

Thus, the nilpotent symmetry transformations $\tilde{s}_{(a)b}$ do not fulfill one of the central criteria of the BRST formalism. To obtain the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations, we shall take recourse to the superfield formalism in the next section.

4. Off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations: superfield formalism

It is clear, from our earlier discussions, that the celebrated 4D topological term (i.e. $(m/4)\varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} F_{\eta\kappa}$) is a gauge (and, therefore, (anti-) BRST) invariant quantity. As a consequence, for all practical purposes, the Lagrangian density $\mathcal{L}_0$ can be treated as the sum of the *free* Abelian 1-form and 2-form gauge theories (whose nilpotent symmetries we are going to discuss below).

The off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations can be derived by exploiting the standard techniques of the superfield formalism (see, e.g. [30,31] and [36-38] for details). For this paper to be self-contained, we provide *firstly* a very concise description of the superfield formalism, applied to the case of Abelian 1-form gauge theory [36-38] (later on, we shall provide the superfield description of 2-form theory). In this context, it is worthwhile to point out that the curvature tensor $F_{\mu\nu}$, owing its origin to the exterior derivative d (i.e. $dA^{(1)} = (1/2!)(dx^\mu \wedge dx^\nu) F_{\mu\nu}$), remains invariant under the (anti-) BRST symmetry transformations. This observation remains intact as we proceed ahead from the ordinary 4D field theory to the superfield formalism on the (4, 2)-dimensional supermanifold. Thus, first of all, we generalize (in our superfield formalism) the exterior derivative d to its counterpart on the (4, 2)-dimensional supermanifold as

$$d \longrightarrow \tilde{d}\ =\ dZ^M \partial_M\ \equiv\ dx^\mu\, \partial_\mu + d\theta\, \partial_\theta + d\bar{\theta}\, \partial_{\bar{\theta}}, \tag{19}$$

where $Z^M = (x^\mu, \theta, \bar{\theta})$, $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$ are the superspace variables and corresponding partial derivatives on the (4, 2)-dimensional supermanifold.

Here the bosonic spacetime variables $x^\mu (\mu = 0, 1, 2, 3)$ and a pair of Grassmannian variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) parametrize the above supermanifold. After this, we generalize the basic fields $(A_\mu, C, \bar{C})$, defined on the 4D ordinary spacetime Minkowski manifold, to the corresponding superfields (defined on the (4, 2)-dimensional supermanifold) with the following expansions along the Grassmannian directions (see, e.g. [30-39])

$$\begin{aligned} \mathcal{B}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta\, \bar{R}_\mu(x) + \bar{\theta}\, R_\mu(x) + i\, \theta\, \bar{\theta}\, S_\mu(x), \\ \mathcal{F}(x, \theta, \bar{\theta}) &= C(x) + i\, \theta\, \bar{b}_1(x) + i\, \bar{\theta}\, b_2(x) + i\, \theta\, \bar{\theta}\, s(x), \\ \bar{\mathcal{F}}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i\, \theta\, \bar{b}_2(x) + i\, \bar{\theta}\, b_1(x) + i\, \theta\, \bar{\theta}\, \bar{s}(x), \end{aligned} \tag{20}$$

where, on the r.h.s., the fields $(R_\mu, \bar{R}_\mu, s, \bar{s})$ and $(S_\mu, b_1, \bar{b}_1, b_2, \bar{b}_2)$ are the fermionic and bosonic secondary fields, respectively. These fields can be expressed in terms of the basic and auxiliary fields of the theory when we exploit the potential of the horizontality condition (HC).

The celebrated HC requires that the 2-form super-curvature should be equated with the ordinary 2-form curvature as follows

$$\tilde{d}\, \tilde{A}^{(1)} = d\, A^{(1)} \;\Rightarrow \tilde{F}_{\mu\nu}(x, \theta, \bar{\theta}) = F_{\mu\nu}(x), \tag{21}$$

where the super 1-form connection $\tilde{A}^{(1)}$ is defined, in terms of multiplet superfields $(\mathcal{B}_\mu(x, \theta, \bar{\theta}), \mathcal{F}(x, \theta, \bar{\theta}), \bar{\mathcal{F}}(x, \theta, \bar{\theta}))$, as given below

$$\tilde{A}^{(1)} = dZ^M A_M \equiv dx^\mu\, \mathcal{B}_\mu(x, \theta, \bar{\theta}) + d\theta\, \bar{\mathcal{F}}(x, \theta, \bar{\theta}) + d\bar{\theta}\, \mathcal{F}(x, \theta, \bar{\theta}). \tag{22}$$

Furthermore, the HC (cf. (21)) also implies that the super-curvature tensor $\tilde{F}_{\mu\nu}(x, \theta, \bar{\theta})$ is restricted to be equal to the ordinary curvature tensor $F_{\mu\nu}(x)$. The above restriction (i.e. HC) yields the following relationships [39]

$$\begin{aligned} &b_2 = \bar{b}_2 = 0, \qquad s = \bar{s} = 0, \qquad b_1 + \bar{b}_1 = 0, \\ &R_\mu = \partial_\mu C, \qquad \bar{R}_\mu = \partial_\mu \bar{C}, \qquad S_\mu = \partial_\mu B, \end{aligned} \tag{23}$$

where we have chosen the secondary fields b_1 and $\bar{b}_1$ in terms of the Nakanishi-Lautrup auxiliary field B (i.e. $b_1 = B = -\, \bar{b}_1$). The latter is required to linearize the gauge-fixing term (i.e. $B(\partial \cdot A) + (1/2)B^2 = -(1/2)(\partial \cdot A)^2$) in the ordinary (anti-) BRST invariant Lagrangian density (see, Sec. 5 below). Substitution of these fields in the superfield expansions yields the following

$$\begin{aligned} \mathcal{B}^{(h)}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta\, (\partial_\mu \bar{C}(x)) + \bar{\theta}\, (\partial_\mu C(x)) + i\, \theta\, \bar{\theta}\, (\partial_\mu B(x)) \\ &\equiv A_\mu(x) + \theta\, (s_{ab} A_\mu(x)) + \bar{\theta}\, (s_b A_\mu(x)) + \theta\, \bar{\theta}\, (s_b s_{ab} A_\mu(x)), \\ \mathcal{F}^{(h)}(x, \theta, \bar{\theta}) &= C(x) - i\, \theta\, B(x) \;\equiv\; C(x) + \theta\, (s_{ab} C(x)), \\ \bar{\mathcal{F}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i\, \bar{\theta}\, B(x) \;\equiv\; \bar{C}(x) + \bar{\theta}\, (s_b \bar{C}(x)), \end{aligned} \tag{24}$$

where the superscript (h) stands for the superfield expansion after the application of the HC and we have denoted the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations as $s_{(a)b}$[6].

In exactly above fashion, we can *now* generalize the basic fields $B_{\mu\nu}, C_\mu, \bar{C}_\mu, \phi, \beta, \bar{\beta}$ of the ordinary 4D Abelian 2-form gauge theory onto the (4, 2)-dimensional supermanifold and these superfields would have the expansions along the Grassmannian directions as (see, [27] for details)

$$
\begin{aligned}
\mathcal{B}_{\mu\nu}(x,\theta,\bar{\theta}) &= B_{\mu\nu}(x) + \theta\, \bar{R}_{\mu\nu}(x) + \bar{\theta}\, R_{\mu\nu}(x) + i\,\theta\,\bar{\theta}\, S_{\mu\nu}(x),\\
\mathcal{F}_\mu(x,\theta,\bar{\theta}) &= C_\mu(x) + \theta\, \bar{B}^{(1)}_\mu(x) + \bar{\theta}\, B^{(1)}_\mu(x) + i\,\theta\,\bar{\theta}\, f^{(1)}_\mu(x),\\
\bar{\mathcal{F}}_\mu(x,\theta,\bar{\theta}) &= \bar{C}_\mu(x) + \theta\, \bar{B}^{(2)}_\mu(x) + \bar{\theta}\, B^{(2)}_\mu(x) + i\,\theta\,\bar{\theta} \bar{f}^{(2)}_\mu(x),\\
\beta(x,\theta,\bar{\theta}) &= \beta(x) + \theta\, \bar{f}_1(x) + \bar{\theta}\, f_1(x) + i\,\theta\,\bar{\theta}\, b_1(x),\\
\bar{\beta}(x,\theta,\bar{\theta}) &= \bar{\beta}(x) + \theta\, \bar{f}_2(x) + \bar{\theta}\, f_2(x) + i\,\theta\,\bar{\theta}\, b_2(x),\\
\Phi(x,\theta,\bar{\theta}) &= \phi(x) + \theta\, \bar{f}_3(x) + \bar{\theta}\, f_3(x) + i\,\theta\,\bar{\theta}\, b_3(x),
\end{aligned}
\tag{25}
$$

where $(R_{\mu\nu}, \bar{R}_{\mu\nu}, f_1, \bar{f}_1, f_2, \bar{f}_2, f_3, \bar{f}_3, f^{(1)}_\mu, \bar{f}^{(2)}_\mu)$ and $(S_{\mu\nu}, B^{(1)}_\mu, \bar{B}^{(1)}_\mu, B^{(2)}_\mu, \bar{B}^{(2)}_\mu, b_1, b_2, b_3)$ are the fermionic and bosonic set of secondary fields, respectively. In terms of the above superfields, the super 2-form connection on the (4, 2)-dimensional supermanifold can be written as (see, e.g. [27])

$$
\begin{aligned}
\tilde{B}^{(2)} &= \frac{1}{2!}(dx^\mu \wedge dx^\nu)\, \mathcal{B}_{\mu\nu}(x,\theta,\bar{\theta}) \;+\; (dx^\mu \wedge d\theta)\, \bar{\mathcal{F}}_\mu(x,\theta,\bar{\theta})\\
&+ \;(dx^\mu \wedge d\bar{\theta})\, \mathcal{F}_\mu(x,\theta,\bar{\theta}) \;+\; (d\theta \wedge d\theta)\, \bar{\beta}(x,\theta,\bar{\theta})\\
&+ \;(d\bar{\theta} \wedge d\bar{\theta})\, \beta(x,\theta,\bar{\theta}) \;+\; (d\theta \wedge d\bar{\theta})\, \Phi(x,\theta,\bar{\theta}).
\end{aligned}
\tag{26}
$$

The celebrated HC for this system can be expressed as

$$
\tilde{d}\, \tilde{B}^{(2)} \;=\; d\, B^{(2)} \;\Longrightarrow\; \tilde{H}_{\mu\nu\eta}(x,\theta,\bar{\theta}) = H_{\mu\nu\eta}(x).
\tag{27}
$$

In other words, the HC is a restriction such that the super-curvature tensor $\tilde{H}_{\mu\nu\eta}(x,\theta,\bar{\theta})$ is, ultimately, independent of the Grassmannian variables so that $\tilde{H}_{\mu\nu\eta}(x,\theta,\bar{\theta}) = H_{\mu\nu\eta}(x)$. The above condition leads to the following relationships amongst the basic, secondary and auxiliary fields [27]

$$
\begin{aligned}
&b_1 = b_2 = b_3 = 0, \quad f_1 = 0, \quad \bar{f}_2 = 0, \quad \bar{f}_1 + f_3 = 0, \quad f_2 + \bar{f}_3 = 0,\\
&\bar{B}^{(1)}_\mu + B^{(2)}_\mu + \partial_\mu \phi \;=\; 0, \quad B^{(1)}_\mu \;=\; -\,\partial_\mu \beta, \quad \bar{B}^{(2)}_\mu \;=\; -\,\partial_\mu \bar{\beta},\\
&f^{(1)}_\mu \;=\; i\,\partial_\mu f_3 \;\equiv\; -\,i\,\partial_\mu \bar{f}_1, \quad \bar{f}^{(2)}_\mu \;=\; -\,i\,\partial_\mu \bar{f}_3 \;\equiv\; +\,i\,\partial_\mu f_2,\\
&R_{\mu\nu} \;=\; -\,(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad \bar{R}_{\mu\nu} \;=\; -\,(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu),\\
&S_{\mu\nu} \;=\; -\,i\,(\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu) \;\equiv\; -\,i\,(\partial_\mu B_\nu - \partial_\nu B_\mu).
\end{aligned}
\tag{28}
$$

[6]In explicit terms, it can be seen that we have derived: $s_b A_\mu = \partial_\mu C,\ s_b C = 0,\ s_b \bar{C} = iB,\ s_b B = 0$ and $s_{ab} A_\mu = \partial_\mu \bar{C},\ s_{ab} \bar{C} = 0,\ s_{ab} C = -iB,\ s_{ab} B = 0.$

We can make the following choices for the algebraic convenience:

$$
\begin{array}{rclclcrcl}
\bar{f}_3 & = & \rho(x) & = & -f_2(x), & & \bar{B}_\mu^{(1)} & = & \bar{B}_\mu, \\
f_3 & = & \lambda(x) & = & -\bar{f}_1(x), & & B_\mu^{(2)} & = & -B_\mu,
\end{array} \tag{29}
$$

which lead to the derivation of a Curci-Ferrari (CF)-type restriction, in the realm of the Abelian 2-form gauge theory, as:

$$
\bar{B}_\mu^{(1)} + B_\mu^{(2)} + \partial_\mu \phi \; = \; 0 \implies B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0. \tag{30}
$$

The above condition is responsible for the absolute anticommutativity of the (anti-) BRST symmetry transformations as we elaborate below.

After the substitution of the expressions for the secondary fields, the explicit expansions for the superfields are

$$
\begin{array}{rcl}
\mathcal{B}_{\mu\nu}^{(h)}(x,\theta,\bar{\theta}) & = & B_{\mu\nu}(x) - \theta\,(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) - \bar{\theta}\,(\partial_\mu C_\nu - \partial_\nu C_\mu) \\
& + & \theta\,\bar{\theta}\,(\partial_\mu B_\nu - \partial_\nu B_\mu) \\
& \equiv & B_{\mu\nu}(x) + \theta\,(s_{ab}\,B_{\mu\nu}(x)) + \bar{\theta}\,(s_b\,B_{\mu\nu}(x)) \\
& + & \theta\,\bar{\theta}\,(s_b\,s_{ab}\,B_{\mu\nu}(x)), \\
\mathcal{F}_\mu^{(h)}(x,\theta,\bar{\theta}) & = & C_\mu(x) + \theta\,\bar{B}_\mu(x) - \bar{\theta}\,\partial_\mu \beta - \theta\,\bar{\theta}\,\partial_\mu \lambda \\
& \equiv & C_\mu(x) + \theta\,(s_{ab}\,C_\mu(x)) + \bar{\theta}\,(s_b\,C_\mu(x)) \\
& + & \theta\,\bar{\theta}\,(s_b\,s_{ab}\,C_\mu(x)), \\
\bar{\mathcal{F}}_\mu^{(h)}(x,\theta,\bar{\theta}) & = & \bar{C}_\mu(x) - \theta\,\partial_\mu \bar{\beta} - \bar{\theta}\,B_\mu(x) + \theta\,\bar{\theta}\,\partial_\mu \rho \\
& \equiv & \bar{C}_\mu(x) + \theta\,(s_{ab}\,\bar{C}_\mu(x)) + \bar{\theta}\,(s_b\,\bar{C}_\mu(x)) \\
& + & \theta\,\bar{\theta}\,(s_b\,s_{ab}\,\bar{C}_\mu(x)), \\
\Phi^{(h)}(x,\theta,\bar{\theta}) & = & \phi(x) + \theta\,\rho(x) + \bar{\theta}\,\lambda(x) \\
& \equiv & \phi(x) + \theta\,(s_{ab}\,\phi(x)) + \bar{\theta}\,(s_b\,\phi(x)), \\
\beta^{(h)}(x,\theta,\bar{\theta}) & = & \beta(x) - \theta\,\lambda(x) \equiv \beta(x) + \theta\,(s_{ab}\,\beta(x)), \\
\bar{\beta}^{(h)}(x,\theta,\bar{\theta}) & = & \bar{\beta}(x) - \bar{\theta}\,\rho(x) \equiv \bar{\beta}(x) + \bar{\theta}\,(s_b\,\bar{\beta}(x)),
\end{array} \tag{31}
$$

where the superscript (h) denotes the superfield expansion after the application of HC. The above expansions yield the following off-shell nilpotent (anti-) BRST symmetry transformations for the relevant fields of the theory

$$
\begin{array}{l}
s_b B_{\mu\nu} = -\,(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = -\,\partial_\mu \beta, \quad s_b \bar{C}_\mu = -\,B_\mu, \\
s_b \bar{\beta} = -\,\rho, \quad s_b \phi = \lambda, \quad s_b \big[\rho,\ \lambda,\ \beta,\ B_\mu,\ H_{\mu\nu\eta}\big] = 0,
\end{array} \tag{32}
$$

$$
\begin{array}{l}
s_{ab} B_{\mu\nu} = -\,(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = -\,\partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = +\,\bar{B}_\mu, \\
s_{ab} \beta = -\,\lambda, \quad s_{ab} \phi = \rho, \quad s_b \big[\rho,\ \lambda,\ \bar{\beta},\ \bar{B}_\mu,\ H_{\mu\nu\eta}\big] = 0.
\end{array} \tag{33}
$$

The absolute anticommutativity requirement imposes the (anti-) BRST symmetry transformations on the Nakanishi-Lautrup type auxiliary fields as:

$$s_b \bar{B}_\mu = -\partial_\mu \lambda, \quad s_{ab} B_\mu = \partial_\mu \rho. \tag{34}$$

Thus, we have obtained the complete set of (anti-) BRST symmetry transformations in the equations (32), (33) and (34) which are off-shell nilpotent of order two and they are absolutely anticommuting in nature as can be checked from the following explicit example:

$$\{s_b,\ s_{ab}\}\ B_{\mu\nu} = \partial_\mu \big(B_\nu - \bar{B}_\nu\big) - \partial_\nu \big(B_\mu - \bar{B}_\mu\big) = 0. \tag{35}$$

The r.h.s. of the above equation is zero on the constrained surface defined by the equation (30) (which is nothing but the CF-type restriction). For the rest of the fields of the theory, it can be checked that $\{s_b,\ s_{ab}\}\Psi = 0$ for Ψ being the generic field (except $B_{\mu\nu}$ that has been considered in (35)).

5. Nilpotent symmetry invariance: Lagrangian formalism

We begin with the BRST and anti-BRST invariant coupled Lagrangian densities, corresponding to the starting Lagrangian density (1), as

$$\begin{aligned}
\mathcal{L}_B &= -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \frac{1}{12}\, H^{\mu\nu\eta} H_{\mu\nu\eta} + \frac{1}{4}\, m\, \varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} F_{\eta\kappa} + B(\partial \cdot A) \\
&+ \frac{1}{2}\, B^2 + B^\mu (\partial^\nu B_{\nu\mu} - \partial_\mu \phi) + B \cdot B - i\, \partial_\mu \bar{C}\, \partial^\mu C + \partial_\mu \bar{\beta}\, \partial^\mu \beta \\
&+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)\, (\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda,
\end{aligned} \tag{36}$$

$$\begin{aligned}
\mathcal{L}_{\bar{B}} &= -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \frac{1}{12}\, H^{\mu\nu\eta} H_{\mu\nu\eta} + \frac{1}{4}\, m\, \varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} F_{\eta\kappa} + B(\partial \cdot A) \\
&+ \frac{1}{2}\, B^2 + \bar{B}^\mu (\partial^\nu B_{\nu\mu} + \partial_\mu \phi) + \bar{B} \cdot \bar{B} - i\, \partial_\mu \bar{C}\, \partial^\mu C + \partial_\mu \bar{\beta}\, \partial^\mu \beta \\
&+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)\, (\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda,
\end{aligned} \tag{37}$$

where the scalar field B and vector fields $(B_\mu, \bar{B}_\mu)$ are the Nakanishi-Lautrup type auxiliary fields, the scalar $(\bar{C}, C)$ and vector $(\bar{C}_\mu, C_\mu)$ fields are the fermionic (anti-) ghost fields, $(\bar{\beta}, \beta)$ are the bosonic ghost for ghost fields, $(\rho)\lambda$ are the fermionic auxiliary (anti-) ghost fields and the massless $(\Box \phi = 0)$ scalar field ϕ is required in the gauge-fixing term for the stage-one reducibility (that is present in the second-rank antisymmetric tensor gauge theory).

The Lagrangian density ($\mathcal{L}_B$) respects the following off-shell nilpotent ($s_b^2 = 0$) BRST symmetry[7] transformations (s_b) [27]

$$
\begin{aligned}
& s_b A_\mu = \partial_\mu C, \quad s_b \bar{C} = iB, \quad s_b B_{\mu\nu} = (\partial_\mu C_\nu - \partial_\nu C_\mu), \\
& s_b C_\mu = \partial_\mu \beta, \quad s_b \bar{C}_\mu = B_\mu, \quad s_b \phi = -\lambda, \quad s_b \bar{\beta} = \rho, \\
& s_b [C,\ B,\ \rho,\ \lambda,\ \beta,\ B_\mu,\ H_{\mu\nu\kappa}] = 0,
\end{aligned} \tag{38}
$$

because the above Lagrangian density transforms to a total spacetime derivative as given below:

$$
\begin{aligned}
s_b \mathcal{L}_B &= \partial_\mu \Big[B\, \partial^\mu C + \rho\, \partial^\mu \beta + \lambda\, B^\mu + (\partial^\mu C^\nu - \partial^\nu C^\mu)\, B_\nu \\
&+ m\, \varepsilon^{\mu\nu\eta\kappa} C_\nu\, (\partial_\eta A_\kappa) \Big].
\end{aligned} \tag{39}
$$

As a consequence, the action $S_{(B)} = \int d^4x\, \mathcal{L}_B$ remains invariant under the off-shell nilpotent BRST symmetry transformations (s_b).

The Noether conserved current ($J^\mu_{(B)}$), that emerges due to the continuous BRST symmetry transformations (s_b), is

$$
\begin{aligned}
J^\mu_{(B)} &= (\partial^\mu C^\nu - \partial^\nu C^\mu)\, B_\nu + \frac{1}{2}\, m\, \varepsilon^{\mu\nu\eta\kappa} (\partial_\nu C)\, B_{\eta\kappa} + H^{\mu\nu\eta} (\partial_\nu C_\eta) \\
&+ B\, \partial^\mu C - F^{\mu\nu} (\partial_\nu C) - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)\, (\partial_\nu \beta) + \lambda\, B^\mu \\
&+ \rho\, \partial^\mu \beta - m\, \varepsilon^{\mu\nu\eta\kappa} C_\nu (\partial_\eta A_\kappa).
\end{aligned} \tag{40}
$$

The conservation law ($\partial_\mu J^\mu_{(B)} = 0$) can be proven by exploiting the following equations of motion that emerge from $\mathcal{L}_B$:

$$
\begin{aligned}
& \partial_\mu H^{\mu\nu\eta} + (\partial^\nu B^\eta - \partial^\eta B^\nu) - \frac{1}{2}\, m\, \varepsilon^{\nu\eta\kappa\zeta} F_{\kappa\zeta} = 0, \qquad (\partial \cdot B) = 0, \\
& B_\mu = -\frac{1}{2}\, (\partial^\nu B_{\nu\mu} - \partial_\mu \phi), \quad \partial_\mu F^{\mu\nu} = \partial^\nu B - \frac{1}{2}\, m\, \varepsilon^{\nu\mu\kappa\eta}\, (\partial_\mu B_{\kappa\eta}), \\
& \Box \beta = 0, \quad \Box \bar{\beta} = 0, \quad \Box C = 0, \quad \Box \bar{C} = 0, \quad B = -\, (\partial \cdot A), \\
& \Box \phi = 0, \qquad \lambda = +\, \frac{1}{2}\, (\partial \cdot C), \qquad \rho = -\, \frac{1}{2}\, (\partial \cdot \bar{C}), \\
& \Box C_\mu = \partial_\mu \lambda\ \equiv\ \frac{1}{2}\, \partial_\mu (\partial \cdot C), \quad \Box \bar{C}_\mu = -\partial_\mu \rho\ \equiv\ \frac{1}{2}\, \partial_\mu (\partial \cdot \bar{C}).
\end{aligned} \tag{41}
$$

[7] These transformations and (43) (see below) have been obtained (cf. (32), (33)) by exploiting the superfield approach to BRST formalism in the context of Abelian 2-form gauge theory in our previous section (see, [27] for details). We take here an overall minus sign so that we could be consistent with the transformations in Secs. 2 and 3 for the sake of precise comparison (at least, for the gauge and (anti-) BRST transformations on $B_{\mu\nu}$).

The conserved BRST charge Q_B, corresponding to $J^\mu_{(B)}$ would be given by $Q_B = \int d^3x\ J^0_{(B)}$, whose explicit form is:

$$\begin{aligned} Q_{(B)} &= \int d^3x \Big[B\dot{C} - \dot{B}C + \Pi^{ij}\ (\partial_i C_j - \partial_j C_i) + (\partial^0 C^i - \partial^i C^0) B_i \\ &- (\partial^0 \bar{C}^i - \partial^i \bar{C}^0)\ (\partial_i \beta) + \rho\ \dot{\beta} + \lambda\ B_0 + m\ \epsilon_{ijk}\ C_i\ (\partial_j A_k)\Big]. \end{aligned} \tag{42}$$

It can be checked that it is a conserved ($\dot{Q}_{(B)} = 0$) and nilpotent ($Q^2_{(B)} = 0$). A close look at $Q_{(B)}$ ensures that it is a generalization of the expressions in (7) and (9) (cf. Sec. 2) for $Q_{(gt)}$ and G, respectively.

The Lagrangian density $\mathcal{L}_{\bar{B}}$ respects the following off-shell nilpotent ($s^2_{ab} = 0$) anti-BRST symmetry transformations (s_{ab})

$$\begin{aligned} &s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} C = -iB, \quad s_{ab} B_{\mu\nu} = (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \\ &s_{ab} \bar{C}_\mu = \partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = -\bar{B}_\mu, \quad s_{ab}\phi = -\rho, \quad s_{ab}\beta = \lambda, \\ &s_{ab}[\bar{C},\ B,\ \rho,\ \lambda,\ \bar{\beta},\ \bar{B}_\mu,\ H_{\mu\nu\kappa}] = 0, \end{aligned} \tag{43}$$

because the above $\mathcal{L}_{\bar{B}}$ transforms to a total spacetime derivative[8]

$$\begin{aligned} s_{ab}\mathcal{L}_{\bar{B}} &= \partial_\mu \Big[B\ \partial^\mu \bar{C} - \rho\ \bar{B}^\mu + \lambda\ \partial^\mu \bar{\beta} + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)\ \bar{B}_\nu \\ &+ m\ \varepsilon^{\mu\nu\eta\kappa} \bar{C}_\nu\ (\partial_\eta A_\kappa)\Big]. \end{aligned} \tag{44}$$

As a result, the action ($S_{(\bar{B})} = \int d^4x\ \mathcal{L}_{\bar{B}}$) remains invariant. It should be noted that $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ are equivalent due to $B_\mu - \bar{B}_\mu - \partial_\mu\phi = 0$.

The symmetry invariance under the continuous nilpotent transformations s_{ab} implies a Noether's conserved current as given by

$$\begin{aligned} J^\mu_{(\bar{B})} &= (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)\ \bar{B}_\nu + \frac{1}{2}\ m\ \varepsilon^{\mu\nu\eta\kappa} B_{\nu\eta}\ (\partial_\kappa \bar{C}) + H^{\mu\nu\eta}(\partial_\nu \bar{C}_\eta) \\ &+ B\ \partial^\mu \bar{C} - F^{\mu\nu}(\partial_\nu \bar{C}) + (\partial^\mu C^\nu - \partial^\nu C^\mu)\ (\partial_\nu \bar{\beta}) - \rho\ \bar{B}^\mu \\ &+ \lambda\ \partial^\mu \bar{\beta} - m\ \varepsilon^{\mu\nu\eta\kappa} \bar{C}_\nu (\partial_\eta A_\kappa). \end{aligned} \tag{45}$$

The conservation law ($\partial_\mu J^\mu_{(\bar{B})} = 0$) can be proven by taking into account the equations of motion from $\mathcal{L}_{(\bar{B})}$ that are same as (41) except the following:

$$\begin{aligned} &\bar{B}_\mu = -\frac{1}{2}\ (\partial^\nu B_{\nu\mu} + \partial_\mu \phi), \qquad (\partial \cdot \bar{B}) = 0 \implies \Box\phi = 0, \\ &\partial_\mu H^{\mu\nu\eta}\ +\ (\partial^\nu \bar{B}^\eta\ -\ \partial^\eta \bar{B}^\nu) - \frac{1}{2}\ m\ \varepsilon^{\nu\eta\kappa\zeta} F_{\kappa\zeta} = 0. \end{aligned} \tag{46}$$

---

[8]Under the BRST symmetry transformations s_b (with $s_b \bar{B}_\mu = \partial_\mu \lambda$), the Lagrangian density $\mathcal{L}_{\bar{B}}$ transforms to a total spacetime derivative plus a term that is zero on the constrained surface defined by $B_\mu - \bar{B}_\mu = \partial_\mu \phi$. Exactly, in a similar fashion, $\mathcal{L}_B$ transforms under s_{ab} (with $s_{ab} B_\mu = -\partial_\mu \rho$) to a total spacetime derivative plus a term that is zero on the constrained surface defined by field equation $B_\mu - \bar{B}_\mu = \partial_\mu \phi$.

The corresponding anti-BRST charge ($Q_{(\bar{B})} = \int d^3x \; J^0_{(\bar{B})}$) is

$$
\begin{aligned}
Q_{(\bar{B})} &= \int d^3x \Big[B\dot{\bar{C}} - \dot{B}\bar{C} + \Pi^{ij} \, (\partial_i \bar{C}_j - \partial_j \bar{C}_i) + (\partial^0 C^i - \partial^i C^0)\bar{B}_i \\
&+ \; (\partial^0 C^i - \partial^i C^0) \, (\partial_i \bar{\beta}) - \rho \, \bar{B}_0 + \lambda \, \dot{\bar{\beta}} + m \, \epsilon_{ijk} \, \bar{C}_i \, (\partial_j A_k) \Big]. \qquad (47)
\end{aligned}
$$

The above charge is also a conserved ($\dot{Q}_{(\bar{B})} = 0$) and nilpotent ($Q^2_{(\bar{B})} = 0$). Like $Q_{(B)}$, the anti-BRST charge $Q_{(\bar{B})}$ is also generalization of (7) and (9).

It is worth noting that the equations of motion (41) and (46) imply that the relationship in equation (30) (that corresponds to CF condition) is true. In fact, this constrained field equation defines a surface on the 4D-spacetime manifold where the absolute anticommutativity of the (anti-) BRST symmetries is satisfied. This relationship has also been shown to be connected with the geometrical object called gerbs which are one of the very active areas of research in theoretical high energy physics [40,41]. Now we dwell a bit on the conditions that emerge from the physicality criteria $Q_{(\bar{B})B}| \, phys >= 0$. It can be seen that the conserved and nilpotent BRST charge $Q_{(B)}$ produces

$$
\begin{aligned}
Q_{(B)}| \, phys > = \; 0 \Longrightarrow \Pi^0_{(A)}| \, phys > &= \; 0 \quad \Longrightarrow B| \, phys > = \; 0, \\
\partial_i \Pi^i_{(A)}| \, phys > &= \; 0 \quad \Longrightarrow \dot{B}| \, phys > = \; 0, \\
\Pi^{0i}_{(B)}| \, phys > &= \; 0 \quad \Longrightarrow B^i| \, phys > = \; 0, \\
\partial_i \Pi^{ij}_{(B)}| \, phys > &= \; 0 \quad \Longrightarrow \partial_i H^{0ij}| \, phys > = \; 0. \qquad (48)
\end{aligned}
$$

The same conditions also emerge from the anti-BRST charge $Q_{(\bar{B})}| \, phys > = 0$. The above condition (48) ensure that the BRST quantization method is consistent with the requirements of the Dirac's method of quantization of systems with constraints. Thus, the BRST quantization scheme resolves all the unanswered issues that were raised at the fag end of Sec. 2.

6. (Anti-) BRST invariance: superfield formalism

It is interesting to point out that the coupled Lagrangian densities (36) and (37) can be expressed (modulo some total spacetime derivatives) as

$$
\begin{aligned}
\mathcal{L}_B &= -\frac{1}{4} \, F^{\mu\nu} F_{\mu\nu} + \frac{1}{12} \, H^{\mu\nu\eta} H_{\mu\nu\eta} + \frac{1}{4} \, m \, \varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} F_{\eta\kappa} \\
&+ \; s_b \, s_{ab} \Big[\frac{i}{2} \, A_\mu A^\mu + \frac{1}{2} \, C\bar{C} - \frac{1}{4} \, B_{\mu\nu} B^{\mu\nu} + \bar{C}_\mu C^\mu + 2\beta\bar{\beta} \Big], \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\bar{B}} &= -\frac{1}{4} \, F^{\mu\nu} F_{\mu\nu} + \frac{1}{12} \, H^{\mu\nu\eta} H_{\mu\nu\eta} + \frac{1}{4} \, m \, \varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} F_{\eta\kappa} \\
&- \; s_{ab} \, s_b \Big[\frac{i}{2} \, A_\mu A^\mu + \frac{1}{2} \, C\bar{C} - \frac{1}{4} \, B_{\mu\nu} B^{\mu\nu} + \bar{C}_\mu C^\mu + 2\beta\bar{\beta} \Big], \qquad (50)
\end{aligned}
$$

where we have to exploit the (anti-) BRST transformations quoted in (43) and (38). Furthermore, we have to tap the usefulness of the CF-type restriction (that is written in (30)) so that $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ can be expressed in the particular forms that are quoted in equations (36) and (37).

It is evident from the super expansion (24) and (31) that the (anti-) BRST symmetry transformations for a 4D ordinary field can be expressed in terms of the translations of the corresponding superfield along the Grassmannian directions of the (4, 2)-dimensional supermanifold, as[9]

$$s_b \Omega(x) = \lim_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \, \Omega^{(h)}(x, \theta, \bar{\theta}), \quad s_{ab} \Omega(x) = \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \theta} \, \Omega^{(h)}(x, \theta, \bar{\theta}), \tag{51}$$

where $\Omega(x)$ is the generic 4D field and $\Omega^{(h)}(x, \theta, \bar{\theta})$ is the corresponding superfield. Furthermore, it is also clear from expansions (24) and (31) that

$$\frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \, \Omega^{(h)}(x, \theta, \bar{\theta}) \; + \; \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \, \Omega^{(h)}(x, \theta, \bar{\theta}) \; = \; 0, \tag{52}$$

which corresponds to the anticommutativity of the (anti-) BRST symmetry transformations in the operator form (cf. (17)). The above expressions provide the geometrical interpretations for the (anti-) BRST symmetry transformations (and their corresponding generators) in the language of the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ (with $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0, \partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0$) along the Grassmannian directions of the (4, 2)-dimensional supermanifold.

We have to recall that the HCs of (21) and (27) imply that

$$\tilde{F}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta}) \; = \; \partial_\mu \, \mathcal{B}^{(h)}_\nu(x, \theta, \bar{\theta}) - \partial_\nu \, \mathcal{B}^{(h)}_\mu(x, \theta, \bar{\theta}) \; = \; F_{\mu\nu}(x), \tag{53}$$

$$\begin{aligned} \tilde{H}^{(h)}_{\mu\nu\eta}(x, \theta, \bar{\theta}) \quad &= \quad \partial_\mu \, \mathcal{B}^{(h)}_{\nu\eta}(x, \theta, \bar{\theta}) + \partial_\nu \, \mathcal{B}^{(h)}_{\eta\mu}(x, \theta, \bar{\theta}) + \partial_\eta \, \mathcal{B}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta}) \\ &= \quad H_{\mu\nu\eta}(x), \end{aligned} \tag{54}$$

where $\tilde{F}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta})$ and $\tilde{H}^{(h)}_{\mu\nu\eta}(x, \theta, \bar{\theta})$ are the super-curvature tensors after the application of the HC. The above conditions (53) and (54) show that $\tilde{F}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta})$ and $\tilde{H}^{(h)}_{\mu\nu\eta}(x, \theta, \bar{\theta})$ are basically independent of the Grassmannian variables which can be proven by taking into account (24) and (31). Thus, the kinetic term of the Lagrangian densities (49) and (50) can be written in terms of the superfields (after the application of HC) as:

$$- \; \frac{1}{4} \, \tilde{F}^{\mu\nu(h)}(x, \theta, \bar{\theta}) \; \tilde{F}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta}) + \frac{1}{12} \, \tilde{H}^{\mu\nu\eta(h)}(x, \theta, \bar{\theta}) \; \tilde{H}^{(h)}_{\mu\nu\eta}(x, \theta, \bar{\theta}), \tag{55}$$

[9]It should be noted that there is an overall sign difference between the transformations ((32), (33)) and ((38), (43)). Thus, the mapping quoted below (cf. (51)) is correct modulo the sign factor in the context of (anti-) BRST symmetries for the Abelian 2-form theory.

which are actually independent of Grassmannian variables. Thus, without the inclusion of the topological $\big[T(x) = (B \wedge F)(x)\big]$ term, we can express the rest part of the Lagrangian densities (49) and (50), in the language of superfields (obtained after the application of HC), as

$$\begin{aligned}\tilde{\mathcal{L}}_B &= -\frac{1}{4}\,\tilde{F}^{(h)}_{\mu\nu}\,\tilde{F}^{\mu\nu(h)} + \frac{1}{12}\,\tilde{H}^{(h)}_{\mu\nu\eta}\,\tilde{H}^{\mu\nu\eta(h)} + \frac{\partial}{\partial\bar{\theta}}\,\frac{\partial}{\partial\theta}\Big[\frac{i}{2}\,\mathcal{B}^{(h)}_{\mu}\,\mathcal{B}^{\mu(h)} \\ &+ \frac{1}{2}\,\mathcal{F}^{(h)}\,\bar{\mathcal{F}}^{(h)} - \frac{1}{4}\,\mathcal{B}^{(h)}_{\mu\nu}\,\mathcal{B}^{\mu\nu(h)} + \bar{\mathcal{F}}^{(h)}_{\mu}\mathcal{F}^{\mu(h)} + 2\,\beta^{(h)}\,\bar{\beta}^{(h)}\Big], \qquad (56)\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{L}}_{\bar{B}} &= -\frac{1}{4}\,\tilde{F}^{(h)}_{\mu\nu}\,\tilde{F}^{\mu\nu(h)} + \frac{1}{12}\,\tilde{H}^{(h)}_{\mu\nu\eta}\,\tilde{H}^{\mu\nu\eta(h)} - \frac{\partial}{\partial\theta}\,\frac{\partial}{\partial\bar{\theta}}\Big[\frac{i}{2}\,\mathcal{B}^{(h)}_{\mu}\,\mathcal{B}^{\mu(h)} \\ &+ \frac{1}{2}\,\mathcal{F}^{(h)}\,\bar{\mathcal{F}}^{(h)} - \frac{1}{4}\,\mathcal{B}^{(h)}_{\mu\nu}\,\mathcal{B}^{\mu\nu(h)} + \bar{\mathcal{F}}^{(h)}_{\mu}\mathcal{F}^{\mu(h)} + 2\,\beta^{(h)}\,\bar{\beta}^{(h)}\Big], \qquad (57)\end{aligned}$$

where $\tilde{\mathcal{L}}_B$ and $\tilde{\mathcal{L}}_{\bar{B}}$ are the super Lagrangian densities defined on the (4, 2)-dimensional supermanifold (without the topological term). It is elementary now to note that the above super Lagrangian densities satisfy

$$\begin{aligned}&\lim_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\,\tilde{\mathcal{L}}_B = 0, \qquad \lim_{\bar{\theta}\to 0}\frac{\partial}{\partial\theta}\,\tilde{\mathcal{L}}_B = 0, \\ &\lim_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\,\tilde{\mathcal{L}}_{\bar{B}} = 0, \qquad \lim_{\bar{\theta}\to 0}\frac{\partial}{\partial\theta}\,\tilde{\mathcal{L}}_{\bar{B}} = 0, \qquad (58)\end{aligned}$$

which capture the (anti-) BRST invariance of the Lagrangian densities (49) and (50) (without $T(x)$ term) in the physical four dimensions of spacetime.

Now we focus on the (anti-) BRST invariance of the topological $\big[T(x) = (B\wedge F)(x)\big]$ term of the 4D Lagrangian densities (49) and (50). The superfield generalization of this term is given below:

$$T(x) \to \tilde{T}(x,\theta,\bar{\theta}) = \frac{1}{4}\,m\,\varepsilon^{\mu\nu\eta\kappa}\,\mathcal{B}^{(h)}_{\mu\nu}(x,\theta,\bar{\theta})\,\tilde{\mathcal{F}}^{(h)}_{\eta\kappa}(x,\theta,\bar{\theta}). \qquad (59)$$

It is clear from the HC for the 1-form gauge theory (cf. (21)) that $\tilde{F}^{(h)}_{\mu\nu}(x,\theta,\bar{\theta}) = F_{\mu\nu}(x)$. Thus, the above topological term can be expressed in terms of superfields (after the application of HC) as[10]

$$\begin{aligned}\tilde{T}(x,\theta,\bar{\theta}) &= \frac{1}{4}\,m\,\varepsilon^{\mu\nu\eta\kappa}\Big[B_{\mu\nu}(x) + \theta\,(\partial_\mu\bar{C}_\nu - \partial_\nu\bar{C}_\mu) + \bar{\theta}\,(\partial_\mu C_\nu - \partial_\nu C_\mu) \\ &+ \theta\,\bar{\theta}\,(\partial_\mu B_\nu - \partial_\nu B_\mu)\Big]\,F_{\eta\kappa}(x). \qquad (60)\end{aligned}$$

[10]Note that we have taken here the positive signs in the expansion of $\mathcal{B}^{(h)}_{\mu\nu}(x,\theta,\bar{\theta})$ to be consistent with our transformations in (38) and (43).

The BRST, anti-BRST and combined (anti-) BRST invariance of the above term can be expressed, in the language of the superfield formalism, as

$$\begin{aligned}
&\lim_{\theta\to 0}\frac{\partial}{\partial\bar\theta}\,\tilde T(x,\theta,\bar\theta) = \partial_\mu\,\big[m\,\varepsilon^{\mu\nu\eta\kappa} C_\nu\,\partial_\eta A_\kappa\big] \equiv s_b\,(T(x)),\\
&\lim_{\bar\theta\to 0}\frac{\partial}{\partial\theta}\,\tilde T(x,\theta,\bar\theta) = \partial_\mu\,\big[m\,\varepsilon^{\mu\nu\eta\kappa} \bar C_\nu\,\partial_\eta A_\kappa\big] \equiv s_{ab}\,(T(x)),\\
&\frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\tilde T(x,\theta,\bar\theta) = \partial_\mu\,\big[m\,\varepsilon^{\mu\nu\eta\kappa} B_\nu\,\partial_\eta A_\kappa\big] \equiv s_b\,s_{ab}\,(T(x)).
\end{aligned}\tag{61}$$

The above equations imply that the topological term transforms to the total spacetime derivatives under BRST, anti-BRST and combined (anti-) BRST symmetry transformations. As a consequence, the action remains invariant under the nilpotent (anti-) BRST symmetry transformations.

It is worthwhile to point out that the topological term is somewhat different from the rest of the terms of the Lagrangian densities (36) and (37) because it always transforms to a total spacetime derivative under the gauge and (anti-) BRST symmetry transformations. This is what is reflected in (61) within the framework of geometrical superfield formalism. It is clear from equation (58) that the super Lagrangian densities (without the topological term) are such that their translations along the Grassmannian directions lead to zero result. The super-topological term (60), however, behaves in a distinct manner because its translation along the Grassmannian directions (i.e. $\theta, \bar\theta$ and $\theta\bar\theta$) lead always to a total spacetime derivative term (cf. (61)).

7. Conclusions

We have performed the BRST quantization of the 4D topological massive Abelian $U(1)$ gauge model (in the presence of the celebrated $B \wedge F$ term). Our (anti-) BRST symmetry transformations (38) and (43) respect a couple of basic requirements of the BRST formalism because they satisfy

(i) the off-shell nilpotency of order two ($s^2_{(a)b} = 0$), and

(ii) the absolute anticommutativity property ($s_b s_{ab} + s_{ab} s_b = 0$) on the constrained surface defined by the field equation (30).

It is the superfield formalism, proposed in [30-39], that has been able to help us in achieving the above type of (anti-) BRST symmetry transformations that obey the basic requirements of the BRST formalism.

One of the key results of our present investigation is the derivation of the CF-type restriction (in the context of the topologically massive Abelian $U(1)$ gauge theory) which enables us to obtain the absolute anticommutativity of the (anti-) BRST symmetry transformations. It should be recalled that, for the first time, the CF condition [25] appeared in the BRST description of the

non-Abelian 1-form gauge theory. In our earlier works [40,41], a deep connection between the CF-type restrictions and the geometrical objects, called gerbes, has been established. In fact, the existence of the CF-type restriction is an inevitable consequence when we exploit the superfield formalism of [30,31] in the context of higher p-form ($p \geq 2$) gauge theories.

The distinguishing feature of the topological term (i.e. $T = B \wedge F$) becomes quite transparent in the framework of superfield formalism. In this connection, mention should be made that the operation of the Grassmannian derivatives on the super-topological expression (59) always yields a total spacetime derivative term. These expressions, in turn, imply the (anti-) BRST invariance of the topological term in 4D. This is not the case, however, with the rest of the terms of the Lagrangian density in 4D (or its counterpart in (4, 2)-dimensional supermanifold) where the (anti-) BRST invariance ensues because of the (anti-) BRST transformations on all the terms.

The central objective of our present investigation has been to take a modest step in the direction to obtain the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations for the topologically massive version of the 4D *non-Abelian* gauge theory. Many attempts [8,28,29], in this direction, have already been made. The existence of the CF-type condition and the coupled Lagrangian densities have not been deduced, however, in the above attempts. Our future endeavor [42] would be to obtain the above mentioned decisive features (in the context of non-Abelian version of our present model) by exploiting the superfield formalism proposed by Bonora, etal. [30,31] to obtain the proper (anti-) BRST symmetries.

Acknowledgements

Financial support from the Department of Science and Technology, Government of India, under the SERC project grant No: SR/S2/HEP-23/2006, is gratefully acknowledged.

References

[1] C. Becchi, A. Rouet, R. Stora, Phys. Lett. B **52**, 344 (1974)

[2] C. Becchi, A. Rouet, R. Stora, Commun. Math. Phys. **42**, 127 (1975)

[3] C. Becchi, A. Rouet, R. Stora, Ann. Phys. (N.Y.) **98**, 287 (1976)

[4] I. V. Tyutin, Lebedev Institute Preprint, Report No: FIAN-39 (1975) (unpublished)

[5] A. Aurilia, Y. Takahashi, Prog. Theor. Phys. **66**, 693 (1981)

[6] T. R. Govindarajan, J. Phys. G **8**, L17 (1982)

[7] T. J. Allen, M. J. Bowick, A. Lahiri, Mod. Phys. Lett. A **6**, 559 (1991)

[8] D. S. Hwang, C. Lee, J. Math. Phys. **38**, 30 (1997)

[9] V. I. Ogievetsky, I. V. Palubarinov, Yad. Fiz. **4**, 216 (1966)

[10] V. I. Ogievetsky, I. V. Palubarinov, Sov. J. Nucl. Phys. **4**, 156 (1967)

[11] M. B. Green, J. H. Schwarz, E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987)

[12] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, 1998)

[13] A. Salam, E. Sezgin (eds.), in *Supergravities in Diverse Dimensions* (World Scientific, Singapore, 1989)

[14] E. Cartan, Compt. Rend. **182**, 956 (1926)

[15] E. Harikumar, R. P. Malik, M. Sivakumar, J. Phys. A: Math. Gen. **33**, 7149 (2000). hep-th/0004145
R. P. Malik, J. Phys. A: Math. Gen. **36**, 5095 (2003). hep-th/0209136

[16] S. Gupta, R. Kumar, R. P. Malik, Eur. Phys. J. C **65**, 311 (2010). arXiv: 0905.0934 [hep-th]

[17] Saurabh Gupta, R. P. Malik, Eur. Phys. J. C **58**, 517 (2008). arXiv: 0807.2306 [hep-th]

[18] B. Julia, G. Toulouse, J. de Phys. **16**, 395 (1979)

[19] See, e.g., R. K. Kaul, Phys. Rev. D **18**, 1127 (1978)

[20] See, e.g., P. K. Townsend, Phys. Lett. B **88**, 97 (1979)

[21] See, e.g., H. Hata, T. Kugo, N. Ohta, Nucl. Phys. B **178**, 527 (1981)

[22] See, e.g., T. Kimura, Prog. Theor. Phys. **64**, 357 (1980)

[23] K. Nishijima, Czech. J. Phys. **46**, 1 (1996)

[24] N. Nakanishi, I. Ojima, *Covariant Operator Formalism of Gauge Theory and Quantum Gravity* (World Scientific, Singapore, 1990)

[25] G. Curci, R. Ferrari, Phys. Lett. B **63**, 91 (1976)

[26] A. Lahiri, Phys. Rev. D **55**, 5045 (1997)

[27] R. P. Malik, Eur. Phys. J. C **60**, 457 (2009). hep-th/0702039
R. P. Malik, arXiv: 0912.4871 [hep-th]

[28] J. Thierry-Mieg, L. Baulieu, Nucl. Phys. B **228**, 259 (1983)

[29] D. Z. Freedman, P. K. Townsend, Nucl. Phys. B **177**, 282 (1981)

[30] L. Bonora, M. Tonin, Phys. Lett. B **98**, 48 (1981)

[31] L. Bonora, P. Pasti, M. Tonin, Nuovo Cim. A **63**, 353 (1981)

[32] S. Weinberg, *The Quantum Theory of Fields: Modern Applications*, vol. 2 (Cambridge University Press, Cambridge, 1996)

[33] P. A. M. Dirac, *Lectures on Quantum Mechanics, Belfer Graduate School of Science* (Yeshiva University Press, New York, 1964)

[34] K. Sundermeyer, *Constrained Dynamics: Lecture Note in Physics*, vol. 169 (Springer-Verlag, Berlin, 1982)

[35] See, e.g., P. Mitra, R. Rajaraman, Ann. Phys. (N.Y.) **203**, 137 (1990)

[36] R. P. Malik, Ann. Phys. (N. Y.) **307**, 01 (2003). hep-th/0205135

[37] See, e.g., R. P. Malik, Phys. Lett. B **584**, 210 (2004). hep-th/0311001

[38] R. P. Malik, J. Phys. A: Math. Gen. **37**, 5261 (2004). hep-th/0311193

[39] See, e.g., R. P. Malik, Int. J. Mod. Phys. A **23**, 3685 (2008). arXiv: 0704.0064 [hep-th]

[40] L. Bonora, R. P. Malik, Phys. Lett. B **655**, 75 (2007). arXiv: 0707.3922 [hep-th]

[41] L. Bonora, R. P. Malik, J. Phys. A: Math. Theor. **43**, 375403 (2010). arXiv: 0911.4919 [hep-th]

[42] S. Gupta, R. Kumar, R. P. Malik, in preparation